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Scale invariance in non-relativistic quantum mechanics

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Scale transformations involving space and time stretching are quite useful in classical mechanics. In non-relativistic quantum mechanics, however, such a transformation leaves the equation of motion invariant only for the inverse cube law of force. Nevertheless extending the concept of scale transformations to include changes in the coupling strength we are able to obtain useful results for any power law potential. In particular the virial theorem can be proven and the eigenfunctions can be shown to be of canonical dimension.

1. INTRODUCTION

Scale transformations have attracted considerable attention in recent years. The Bjorken scaling behaviour of the form factors in deep inelastic electron proton scattering (Bjorken 1969), the Feynman scaling for inclusive reactions (Feynman 1969) and Yang's hypothesis of limiting fragmentotion (Yang 1969) seem to indicate scale invariant behaviour at high energies. Again the idea of scale invariance has found applications in the study of critical phenomena variance has found applications in study of critical phenomena (Widom 1965). However, despite the efficacy of scale transformations in the systematisation of various observations in a wide realm of physical phenomena, the connection between such transformations and the dynamics of various systems needs amplification.

In classical mechanics, the invariance of the Newton's equation of motion for a partical moving in a power law potential under the transformation

$$x' = \lambda x \qquad \ldots \quad (1a)$$

$$t' = \lambda^{1-k/2}t \qquad \ldots \quad (1b)$$

leads to the well known principle of mechanical similarity (Landau & Lifshitz 1966) from which follows, for example, for a free particle ($k = 0$) the linearity with time of the displacement for a simple harmonic oscillator ($k = 2$) the independence of the period on the amplitude, and for the inverse square force

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($k = -1$) Kepler's third law of planetary motion Also the form invariance of the Lagrangian under this scale transformation leads to a conserved quantity

$$D = Et + \frac{1}{k}\boldsymbol{x}\,\boldsymbol{p} - \frac{k+2}{k}\int \tfrac{1}{2}mv^2 dt \qquad .. \quad (2)$$

where E, $\boldsymbol{p}$, m, v are the energy, momentum, mass and velocity of the particle respectively From the conservation law

$$\frac{dD}{dt} = 0 \qquad ... \quad (3)$$

the virial theorem

$$\langle T \rangle = \frac{k}{2}\langle V \rangle \qquad ... \quad (4)$$

relating the average kinetic and potential energy, may be proven.

In non-relativistic quantum mechanics, on the other hand, the Schroedinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\Psi(\boldsymbol{r}, t) = i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{r}, t) \qquad ... \quad (5)$$

for power law potential cannot be kept invariant by scale transformations on space and time alone, unless one is concerned with force-free motion (which is trivial) or the inverse cube law of force (which is unphysical, (Landau & Lifschitz 1966) because the orbiting particle under such a force *falls into the centre*). In relativistic quantum mechanics of particles, on the other hand, as an examination of the Dirac or Klein-Gordon equation shows such transformations may be of relevance (Jackiw 1972) for the physically interesting $1/r$ potential (and even the Yukawa potential as far as short distance behaviour is concerned)

In order to overcome this limitation, we have in the present paper, relaxed the requirement of transforming only space and time but also scale the coupling constant suitably to maintain the invariance of the Schroedinger equation for an arbitrary power law potential. This enables us to obtain a constant of the motion (which we discuss for various potentials) and to prove the virial theorem.

2 Scale Transformation and the Schroedinger Equation for Power Law Potentials

To begin with, it is important to note that the scale transformation considered should not be confused with the independent dimensional scaling of quantities like space, time and mass. It is obvious that the laws of motions are invariant under the choice of the units for these quantities since this type of choice of units are independent of each other.

But the scale transformations is of marked difference since it is done here for canonically conjugate quantities like space and momentum in a mutually connected way. Here the transformation is such that in quantum mechanics the canonical commutation relations remain intact, it is clear that

$$x \rightarrow x' = \lambda x \qquad \ldots (6a)$$

$$p \rightarrow p' = p/\lambda. \qquad \ldots (6b)$$

Thus it follows that the Hamiltonian for a power law potential

$$H = \frac{p^2}{2m} + gr^k \qquad \ldots (7)$$

can be kept form invariant under scale transformations if we admit a transformation of the coupling constant (strength of the potential)

$$g \rightarrow g' = g/\lambda^{k+2} \qquad \ldots (8)$$

and accordingly the Hamiltonian transforms, keeping its form intact, as

$$H \rightarrow H' = H/\lambda^2 \qquad \ldots (9)$$

The Lagrangian transforms in the same manner as the Hamiltonian.

Thus under an infinitesimal scale transformation

$$\delta L = \frac{\partial L}{\partial x_j} \delta x_j + \frac{\partial L}{\partial \dot{x}_j} \delta \dot{x}_j + \frac{\partial L}{\partial g} \delta g \qquad \ldots (10)$$

Using eqs (6) to (9) and the relations $H = -L + p_j \dot{x}_j$ and $L = T - V$, we obtain

$$2H - \frac{d}{dt}(x_j p_j) - (k+2)V = 0 \qquad \ldots (11)$$

which corresponds to a conserved quantity given by

$$kHt + x_j p_j - (k+2) \int T dt = k \left[Ht + \frac{1}{k} x_j p_j - \frac{k+2}{k} \int \tfrac{1}{2} m v^2 dt \right] \qquad \ldots (12)$$

which is in fact, apart from a factor of k, nothing but the conserved quantity given by eq. (2). Thus while the Schroedinger equation forced us to transform the coupling constant in contrast to the situation with the Newton's equation, the constant of the motion in both cases is the same, namely, D as given by eq (2) and from the corresponding conservation law (eq. 3) the virial theorem (eq. 4) follows in both cases remembering to replace orbit everages by expectation values in stationary states (in view of Ehrenfest's theorem).

To find the infinitesimal generator of scale transformations D, the dilatation operator, we note that from eqs. (6) to (9) follows

$$[D, x_j] = x_j \quad \text{... (13a)}$$

$$[D, g] = -(k+2)g \quad \text{... (13b)}$$

$$[D, H] = -2H. \quad \text{... (13c)}$$

Thus the operator

$$D = \frac{2tH}{\hbar} + i\left\{x_j \frac{\partial}{\partial x_j} - (k+2)g\frac{\partial}{\partial g}\right\} \quad \text{... (14)}$$

satisfies the required commutation relations (eq 13) and is a constant of the motion in the sense that its tot time derivative vanishes namely,

$$) = -i[D, H] + \frac{\partial D}{\partial t} = 0 \quad \text{... (15)}$$

Now it is useful to explore the dependence of Energy of particle of mass m of such a system (i.e., in this form of potential $V = gr^k$) on the coupling strength g

Again it is quite intuitive to write E as an unknown function of m, g and $\hbar$ (where $\hbar = 1/2\pi \times$ Planck's constant).

$$E = f(m, g, \hbar) = m^a, g^b, \hbar^c \quad \text{... (16)}$$

where a, b and c are constants.

$$(M)^a(EL^{-k})^b(ET)^c = ML^2T^{-2} \quad \text{... (17)}$$

And it follows that

$$b = \frac{2}{2+k} \quad \text{... (18)}$$

Therefore,

$$E \sim g^{2/2+k}. \quad \text{... (19)}$$

This yields the nature of dependance of energy on the coupling constant. The solution of the Schroedinger equation (eq. 5) for a stationary state is in the form

$$\Psi(\mathbf{r}, t) = e^{\frac{-iEt}{\hbar}} \Psi(\mathbf{r}) \quad \text{... (20)}$$

and it is readily found that

$$i(k+2)g\frac{\partial}{\partial g} e^{\frac{-iEt}{\hbar}} = \frac{2Et}{\hbar} e^{\frac{-iEt}{\hbar}} \quad \text{... (21)}$$

Since $E \sim g^{2/2+k}$.

Using eq. (14) for the definition of D and eq. (15) we have

$$D\Psi(\boldsymbol{r}, t) = \left[\frac{2tH}{\hbar} + i\left\{ x_j \frac{\partial}{\partial x_j} - (k+2)g\frac{\partial}{\partial g} \right\} \right] e^{\frac{-iEt}{\hbar}} \Psi(\boldsymbol{r}), \quad \ldots \quad (22)$$

or,

$$D\Psi(\boldsymbol{r}, t) = i\left\{ \left[x_j \frac{\partial}{\partial x_j} - (k+2)g \frac{\partial}{\partial g} \right] \Psi(\boldsymbol{r},) \right\} e^{\frac{-iEt}{\hbar}} \quad \ldots \quad (23)$$

Since,

$$\left\{ \left[-i(k+2)g \frac{\partial}{\partial g} + \frac{2Et}{\hbar} \right] e^{\frac{-iEt}{\hbar}} \right\} \quad \ldots \quad (24)$$

which follows from eq. (21)

We now define the reduced dilation generator $\mathfrak{D}$ in the form

$$\mathfrak{D} = x_j \frac{\partial}{\partial x_j} - (k+2)g\frac{\partial}{\partial g} = r\frac{\partial}{\partial r} - (k+2)g\frac{\partial}{\partial g} \quad \ldots \quad (25)$$

It is quite striking a result that the reduced dilatation generator is an eigeno-operator for the time independent part of the wave functions with the most interesting power law potential having $k = -1, 2$

The wave function for the potential ($V = gr^{-1}$) is given by (Messiah 1965)

$$\Psi_{k=-1}(r) \simeq r^l(-\alpha g)^{\,l+3/2} e^{-1/n(-2\alpha g r)} L^{2l+1}_{n-l-1}\left(-\frac{rg\alpha}{n} \right) \quad \ldots \quad (26)$$

where

$$V = -\frac{Ze^2}{r} = \frac{g}{r} \quad \text{and} \quad E = -\alpha g^2;\ \alpha = \frac{2\pi m}{h^2 n^2}$$

And its follows that

$$\mathfrak{D}\Psi_{k=-1}(r) = \left(r\frac{\partial}{\partial r} - g\frac{\partial}{\partial g} \right) \Psi_{k=-1}(r) = -\frac{3}{2}\Psi_{k=-1}(r). \quad \ldots \quad (27)$$

Similarly for the 3-dimensional Harmonic Oscillator ($V = gr^2$) the wave function is given by (D. Shallir-Talmi; Nuclear Shell theory)

$$\Psi_{k=2}(r) \simeq g^{\,l/4+3/8} \exp -\left\{ \sqrt{\frac{gm}{2\hbar^2}}\, r^2 \right\} r^{l+1} L^{l+\frac{1}{2}}_{n+l-\frac{1}{2}}\left(\sqrt{\frac{2gm}{\hbar^2}} r^2 \right). \quad \ldots \quad (28)$$

And it is not difficult to show that

$$\mathfrak{D}\Psi_{k=2}(r) = \left(r\frac{\partial}{\partial r} - 4g\frac{\partial}{\partial g} \right) \Psi_{k=2}(r) = -\frac{3}{2}\Psi_{k=2}(r). \quad \ldots \quad (29)$$

Hence the normalised eigenfunction for power law potential with ($k = -1, 2$) are eigenfunctions of the reduced dilation generator

$$\mathcal{D}\Psi = -3/2\Psi \qquad \ldots \quad (30)$$

The eigen value of the reduced dilalation operator is ($-3/2$) Again the dimension assigned to the wave-function Ψ is $L^{-3/2}$, this is a cannonical demension since (it follows from the normalisation condition $\int \Psi^*\Psi d^3X = 1$)

And it is interesting to find that the reduced dilation operator (following from scale transformation) yields the cannonical dimension (following from usual dimensional analysis) of the wave function for the power law potentials.

3. Conclusion

We have argued that scale transformations involving space and time alone, though quite useful in classical mechanics, are expected to be of interest in non-relativistic quantum mechanics only for the inverse cube law of force. Nevertheless if we are prepared to extend the concept of scale transformations to include changes in the coupling constant we are able to get results for poker law potentials. In particular we can obtain the virial theorem and the infinitesimol dilatation generator and show that the normalised enegenfunctions of the dilata. tion generator yields canonical dimensions of the wave functionas its eigenvalue.

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